Implementation of Discrete Vector–Valued Wavelettransforms

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Abstract– In this paper the discrete implementation of orthogonal multiwavelet transforms based on 2 scaling functions is discussed. In contrary to one scaling function (singlewavelets) using several scaling functions offers degrees of freedom that can be used to design orthogonal and symmetric, compactly supported wavelets. The discrete implementation of these vector–valued transformations can be executed by a wavelet-like transform using different stages of the filterbank. How to design these particular stages of the filterbank is discussed in this paper. Alternatively to that, it is shown, how to take advantage of the degrees of freedom in designing the two scaling functions in order to construct multiwavelet systems with identical stages.

I. Introduction

In recent years wavelet transforms have gained a lot of interest in many application fields, e.g. image processing [2] or denoising [4]. Most attention was focussed on single wavelet transforms [3], whereby any signal is approximated by dilated and translated versions of the wavelet function $\Psi(t)$. Single wavelet transforms are based on one scaling function $\Phi(t)$ and one wavelet function $\Psi(t)$, which meet the following dilation equations:

$$\Phi(t)=\sum_{k=0}^{n-1} g_k \Phi(2t-k); \qquad \Psi(t)=\sum_{k=0}^{n-1} h_k \Phi(2t-k).$$

The discrete coefficients g_k and h_k appear in the wavelet basis matrix $W=\begin{bmatrix} g_k \\ h_k \end{bmatrix}$, define the discrete transform and allow to construct the continuous basis functions: Starting with $\Phi^0(t)=\delta(t)$ and iterating the left dilation equation results in an arbitrary (discrete) approximation of $\Phi(t)$ and also in the impulse response of the wavelet filters (Figure 1a with $G(z)=\sum_k g_k z^{-k}$, $H(z)=\sum_k h_k z^{-k}$). Symmetry is a desired property, especially for image processing applications [2], however, it is impossible to design (nontrivial) orthogonal and symmetric singlewavelets. Multiwavelets based on several scaling functions offer many degrees of freedom in the design, and allow these properties, for examples see

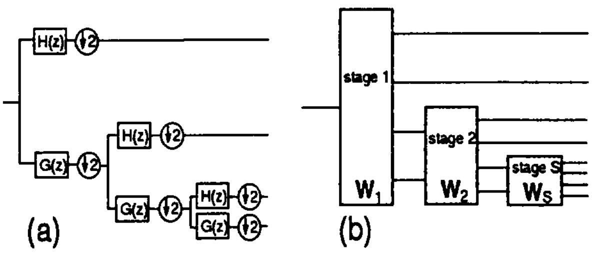

Figure 1: Filterbank implementation of discrete wavelet transforms

[13, 5, 6, 9]. Orthogonal, symmetric multiwavelets also showing a good regularity and frequency behavior are discussed in [11]. Here we focus on the discrete implementation of these vector–valued wavelets using 2 scaling functions and 2 wavelets that are based on 4 dilation equations and the respective basis matrix W of size $4 \times 4m$:

$$\Phi_v=\sum_{l=1}^{2}\sum_{k=0}^{2m-1} g_{v,2k-1+l}\Phi_l(2t-k); v\in\{1,2\}; \quad (1)$$

$$\Psi_v=\sum_{l=1}^{2}\sum_{k=0}^{2m-1} h_{v,2k-1+l}\Phi_l(2t-k); \quad (2)$$

$$W=\begin{pmatrix} \mathbf{W}^U \\ \mathbf{W}^L \end{pmatrix}=\begin{pmatrix} g_{1,0}\; g_{1,1} \cdots g_{1,4m-1} \\ g_{2,0}\; g_{2,1} \cdots g_{2,4m-1} \\ h_{1,0}\; h_{1,1} \ldots h_{1,4m-1} \\ h_{2,0}\; h_{2,1} \ldots h_{2,4m-1} \end{pmatrix}$$

$$W=\begin{pmatrix} \mathbf{A}_1\, \mathbf{A}_2 \ldots \mathbf{A}_m \end{pmatrix} \quad \mathbf{A}_\nu : 4\times 4.$$

In comparison to the singlewavelet case, there are decisive differences in the multiwavelet case with respect to the implementation of the discrete transform. Again the iteration of the dilation equations leads to the continuous bases, however, for the initial step coarse approximations of two scaling functions are required. Only if $\int \Phi_1(t)dt=\int\Phi_2(t)dt$, we have the same procedure as in the case of one scaling function. Starting with $\Phi_1^0(t)=\Phi_2^0(t)=\delta(t)$ and iterating the dilation equations leads to the continuous scaling functions $\Phi_1(t), \Phi_2(t)$, and therefore also to the wavelets. This is the only case, where all stages of the filterbank implementing a vector-valued wavelet transform (see Figure 1b, $W_1 = W_2 = \ldots = W_S$) can be made identical. If both integrals do not vanish, and are not equal, i.e. $\int \Phi_1(t) \neq \int \Phi_2(t)$, the initial functions can be weighted in order to determine the continuous functions. As far as a filterbank implementation is concerned, this results in special prefilters as designed in [14]. If one of the integrals is zero, e.g. $\int \Phi_2(t) = 0$, the proposed discrete transform is a wavelet-like transform [1, 9], where the first stage is completely different from the following stages. The first stage approximates the continuous functions (initial step of iteration). Only the following stages are related to the dilation equations and execute the vector-valued wavelet filtering (the coarse approximation of the previous step is iterated to a finer approximation). In this case the stages W_S of the wavelet-like transform differ as shown in Figure 1b.

This paper shows how to design discrete multiwavelet-like transforms (i.e. its different stages W_S). The proposed method is an algebraic design method requiring solving systems of equations [9]. Alternatively to that, one can take advantage of the degrees of freedom in constructing the scaling functions of a certain multiwavelet system in order to avoid different stages of the transform. This leads to the desired simplicity in design and implementation.

II. Design of Symmetric Multiwavelets

Designing the continuous multiwavelets is equivalent to determine the coefficients of the dilation equations (basismatix W). In [9, 11] it was shown how to get a system of equations for these discrete coefficients representing the properties of the continuous wavelets and scaling functions.

Orthogonality is guaranteed, if the basismatrix fullfills main and shifted orthogonality conditions:

$$\mathbf{W}\mathbf{W}^T = \mathbf{I}; \quad \sum_{i=1}^{j} \mathbf{A}_i \mathbf{A}_{m+i-j}^T = 0, \quad j = 1, 2 \ldots m-1; \tag{3}$$

The equations representing the **approximation** behavior of the wavelet system is formulated for the wavelet coefficients, what guarantees the continuous wavelets fullfilling p vanishing moments $(0 \leq j < p)$.

$$0 = \sum_{l=1}^{2} \sum_{r=0}^{j} \binom{j}{r} I_{l,r} \sum_k h_{v,2k-1+l} k^{j-r}; \quad v \in \{1,2\}; \tag{4}$$

Thereby, the moments of the scaling functions $I_{v,j} = \int t^j \Phi_v(t) dt$ are free parameters and can be computed by the following equations:

$$I_{v,j} = \sum_{l=1}^{2} \sum_{r=0}^{j} \binom{j}{r} I_{l,r} \sum_k g_{v,2k-1+l} k^{j-r}; \quad v \in \{1,2\} \tag{5}$$

In order to design orthogonal and **symmetric**, compactly supported, vector-valued multiwavelets, a specially structured basis matrix W represents the desired symmetry properties: Φ_1 and Ψ_1 are symmetric, Φ_2 and Ψ_2 are antisymmetric.

$$W = \left[g_{1,i}\, g_{2,i}\, h_{1,i}\, h_{2,i}\right]^T =$$

$$= \begin{pmatrix} a_0 & b_0 & a_1 & b_1 & a_1 & -b_1 & a_0 & -b_0 \\ -a_0 & b_0 & a_1 & -b_1 & -a_1 & -b_1 & a_0 & b_0 \\ b_0 & -a_0 & -b_1 & a_1 & -b_1 & -a_1 & b_0 & a_0 \\ b_0 & a_0 & b_1 & a_1 & -b_1 & a_1 & -b_0 & a_0 \end{pmatrix};$$

Solving the resulting system of equations (3–5) allows an approximation order $p = 2$ and leads to the coefficients a_0=0.009977, a_1=0.697129, b_0=b_1=-0.083399 (Thereby, Φ_2 fullfills 1, Ψ_1 fullfills 2 and Ψ_2 has 3 vanishing moments). The corresponding bases are plotted in Figure 2. This approach of de-

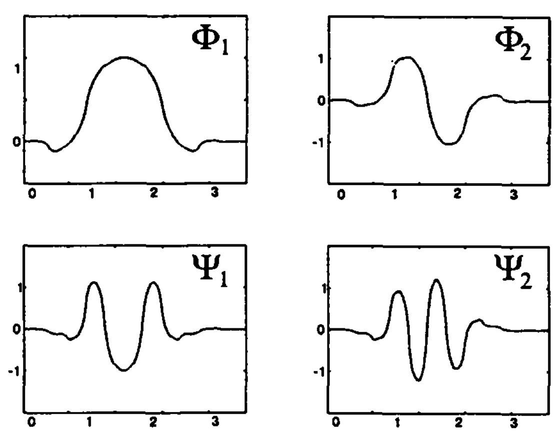

Figure 2: Multiwavelets of $p = 2$ and the corresponding scaling functions

signing orthogonal and symmetric, compactly supported multiwavelets can be extended to arbitrary filterlength leading to arbitrary approximation orders. In the following, we focus on the discrete implementation of these bases, defined by the computed coefficients of the dilation equations.

III. Wavelet-like transforms

For the implementation of the discrete transform, it is essential that $I_{2,0} = 0 = \int \Phi_2(t) dt$, which is given by Φ_2 being antisymmetric. Therefore, a wavelet-like transform with different stages is required. Knowing the properties of the 4 bases (W, section 2), these stages (the basis matrices W_S) are computed separately:

The **first stage** of transform (the rows of $\boldsymbol{W}_1$) should approximate the continuous bases (Figure 1, initial step of iteration) also requiring a special structure (symmetry) of $\boldsymbol{W}_1$:

$$\boldsymbol{W}_1 = \left[g_{1,i}^1\, g_{2,i}^1\, h_{1,i}^1\, h_{2,i}^1\right]^T = \\ = \begin{pmatrix} c_0 & c_1 & c_2 & c_3 & c_3 & c_2 & c_1 & c_0 \\ -c_1 & -c_0 & c_3 & c_2 & -c_2 & -c_3 & c_0 & c_1 \\ -c_1 & c_0 & c_3 & -c_2 & -c_2 & c_3 & c_0 & -c_1 \\ c_0 & -c_1 & c_2 & -c_3 & c_3 & -c_2 & c_1 & -c_0 \end{pmatrix};$$

Of course. $\boldsymbol{W}_1$ must be orthogonal, requiring to fullfill the equations (3). The approximation property of the 4 bases, namely

$$\int \Phi_2(t)dt = 0 \quad \int t^{j_1}\Psi_1(t)dt = 0 \quad \int t^{j_2}\Psi_2(t)dt = 0 \tag{6}$$

($j_1 = \{0,1\}, j_2 = \{0,1,2\}$) consume the remaining degrees of freedom:

$$\sum g_{2,i}^1 = 0 \quad \sum i^{j_1} h_{1,i}^1 = 0 \quad \sum i^{j_2} h_{2,i}^1 = 0. \tag{7}$$

Quite important for the next stage of transform are also the (normalized) moments of the scaling functions $I_{v,j}^1 = i^j g_{v,i}^1, v \in \{1,2\}$.

The task of the **next and all following stages** of transform is to get from the coarse approximation of stage $S-1$ to the next finer one, what is usually executed by the dilation equations. This is the reason for the matrices $\boldsymbol{W}_{S>1}$ being very similar to $\boldsymbol{W}$, and having the same structure as presented in section 2. However, the discrete approximations of the continuous bases of each stage of transform differ, what also leads to different matrices $\boldsymbol{W}_{S>1} = \left[g_{1,i}^S\, g_{2,i}^S\, h_{1,i}^S\, h_{2,i}^S\right]^T$, caused by different values of the parameters $I_{v,j}^{S-1}$.

With respect to computing $\boldsymbol{W}_{S>1}$ orthogonality equations (3) must be fullfilled, as well as the stage–variant approximation equations.

$$0 = \sum_{l=1}^{2}\sum_{r=0}^{j}\binom{j}{r} I_{l,r}^{S-1} \sum_k h_{v,2k-1+l}^S k^{j-r} \quad v \in \{1,2\}; \tag{8}$$

$$I_{v,j}^S = \sum_{l=1}^{2}\sum_{r=0}^{j}\binom{j}{r} I_{l,r}^{S-1} \sum_k g_{v,2k-1+l}^S k^{j-r} \tag{9}$$

This results in a system of equations for each stage of the transform. The solutions guarantee, that the discrete representations of the multiwavelets at each scale are quite accurate and show the desired vanishing moments. Note, the different basis matrices $\boldsymbol{W}_{S>1}$ coverge very fast to the coefficients of the dilation equation ($\boldsymbol{W}$).

IV. Multiwavelets Allowing a Simple Implementation

In section 1 it was shown, that the only way to avoid the different stages of transform is to design multiwavelet systems based on two scaling functions, whose integrals are equal:

$$\int_{-\infty}^{+\infty} \Phi_1(t)dt = \int_{-\infty}^{+\infty} \Phi_2(t)dt. \tag{10}$$

But this condition is only possible, if giving up symmetry of the scaling functions. However, releasing symmetry of the scaling functions does not necessarily cause the loss of symmetry of the wavelets, what is essential (the filter output belonging to the scaling function is further processed in the next stage of the transform, not the wavelet part). There are degrees of freedom in designing the two scaling functions of a certain multiwavelet system in order to fullfill condition (10). It is possible to design two scaling functions, whereby one is the time–reversed of the other allowing a linear combination, namely the wavelets, being symmetric. For the design of this wavelet system, all these assumptions must be embedded into the structure of the wavelet basis matrix:

$$\boldsymbol{W} = \begin{pmatrix} g_{1,i} \\ g_{2,i} \\ h_{1,i} \\ h_{2,i} \end{pmatrix} = \begin{pmatrix} a_0 & b_0 & a_1 & b_1 & a_2 & b_2 & a_3 & b_3 \\ b_3 & a_3 & b_2 & a_2 & b_1 & a_1 & b_0 & a_0 \\ c_0 & d_0 & c_1 & d_1 & d_1 & c_1 & d_0 & c_0 \\ d_0 & c_0 & -d_1 & -c_1 & c_1 & d_1 & -c_0 & -d_0 \end{pmatrix}.$$

Computing the discrete coefficients by the algebraic design method (section 2, [9]) leads to the discrete coefficients $-a_0$=b_0=a_2=-b_2=0.088388, a_1=b_1=0.695880, a_3=b_3=0.011227, c_0=0.070438, d_0=0.054561, c_1=-0.554561, d_1=0.429561 and to the nonsymmetric scaling functions and the symmetric multiwavelets of Figure 3. Note, that the wavelets show the same performance as those of section 2, but *for all stages of transform the same filter coefficients can be used* in order represent the continuous multiwavelets quite well. With respect to an efficient implementation of each stage of this transform, a lattice structure is the method of choice. In Figure 4 ($c_\alpha = \cos\alpha, s_\alpha = \sin\alpha$) the lattice filter is shown implementing the transfer functions $G_v(z) = \sum_{i=0}^{7} g_{v,i} z^{-i}$ and $H_v(z) = \sum_{i=0}^{7} h_{v,i} z^{-i}$ ($v \in \{1,2\}$) of the presented example using only a few 2×2–rotations. Note, that in the domain of these rotations all possible orthogonal multiwavelet solutions of certain length can be parameterized by varying α. In this parameter space, not only the number of vanishing moments can be maximized, but also the regularity or stopband attenuation can be optimized. With respect to a very simple implementation, the same methods can be used as presented in [8, 10] for orthogonal singlewavelets.

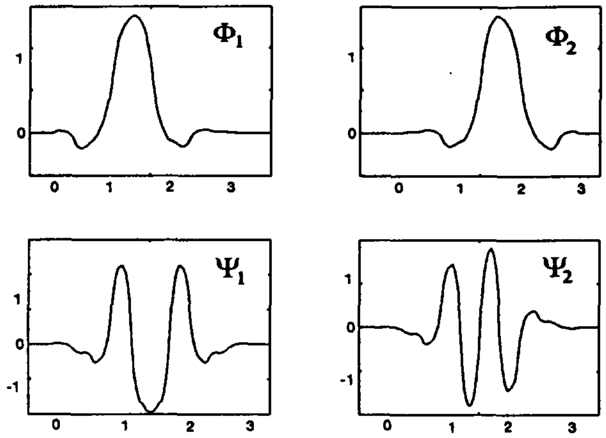

Figure 3: Multiwaveletbases allowing a simple implentation

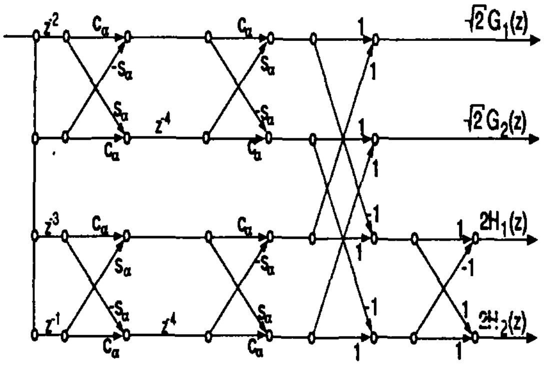

Figure 4: Lattice structure for implementing multiwavelettransforms

V. Conclusion

In this paper the discrete implementation of vector–valued wavelet transforms is discussed. One possibility is to use a wavelet–like transform, requiring different stages of transform. How to design these stages of transforms is shown. Alternatively, special multiwavelet systems are constructed allowing to avoid different stages. For these multiwavelet transforms efficient lattice structures are proposed, that also allow the parameterization of multiwavelet systems of a certain support. The presented approach can be extended to arbitrary lengths and arbitrary approximation orders.

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