APPROXIMATE MOMENTS AND REGULARITY OF EFFICIENTLY IMPLEMENTED ORTHOGONAL WAVELET TRANSFORMS

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ABSTRACT

An efficient implementation of orthogonal wavelet transforms is obtained by approximating the rotation angles of the orthonormal rotations used in a lattice implementation of the filters. This approximation preserves the orthonormality of the transform exactly but leads to non–vanishing moments (except of the zeroth moment). The regularity of these wavelets is analysed by exploiting their finite scale regularity, i.e. "smoothness" only up to a certain finite scale. This finite scale regularity is also related to classical filter banks.

1. INTRODUCTION

In recent years wavelet transforms have gained a lot of interest in various application fields (e.g. [2, 1]). Different kinds of wavelet bases have been designed by posing different conditions on the wavelet/scaling functions. Most attention was paid to orthonormal compactly supported wavelets with a maximal number of vanishing moments [4]. However, other kinds of wavelet bases (more regular [5], biorthogonal wavelets [3]) were also designed. In all these cases the design was based on the convergence and stability of the binary subdivision scheme [14, 15] for scale $j \to \infty$, i.e. the underlaying continuous wavelet/scaling function.

The classical Daubechies' filters of length N were designed to achieve a maximum number of $K = N/2$ vanishing moments. Let h_n be the impulse response of the scaling filter and $g_n = (-1)^n h_{N-1-n}$ the impulse response of the wavelet filter, then the scaling function and the wavelet are defined by

$$\begin{aligned} \phi(t) &= \sum_n h_n \phi(2t-n) \\ \psi(t) &= \sum_n g_n \phi(2t-n)\,. \end{aligned}$$

It was shown in [5] that if the wavelet function $\psi(t)$ of an orthogonal wavelet basis is $K-1$ times continuously differentiable then the wavelet function possesses K vanishing moments. The converse, however, is not true since a wavelet function with K vanishing moments only exhibits a degree of smoothness that asymptotically increases linearly by $\approx 0.2075 \cdot N$ [5]. Recently, this relationship (the gap) between the number of vanishing moments and the actual degree of smoothness has caused various approaches for the design and implementation of wavelet transformations, where the condition of possessing a maximum number of vanishing moments is waived. In [9] it was shown how one can systematically sacrifice higher order vanishing moments to achieve smoother wavelet (scaling) functions. Also, the meassure of smoothness for the design of wavelet basis in the discrete case (when continuous derivatives do not really exist) are discussed in [10] leading to a design of "smooth" wavelets in the discrete domain (i.e. regularity up to a certain scale j of the discrete wavelet transform). The condition of vanishing moments was also sacrificed in [12, 13] in order to achieve a simple implementation of the discrete wavelet transform.

In section 2. we briefly review the efficient implementation of orthonormal wavelet transforms (see [12, 13] for details). Since the moments of these efficiently implemented orthonormal wavelet transforms do not vanish, the decisive question is the influence of approximate moments on the regularity of the wavelet transforms. This is discussed in section 3.

2. EFFICIENTLY IMPLEMENTED ORTHOGONAL WAVELET TRANSFORMS

The efficient implementation is based on the approximation of the rotation angles of elementary 2×2 orthonormal plane rotations using CORDIC–based μ–rotations [8, 7]. Given a representation of an angle β with w–bit accuracy in the basis $\alpha_k = \arctan 2^{-k}$:

$$\beta = \sum_{k=0}^{w} \sigma_k \arctan 2^{-k}\,, \text{ where } \sigma_k \in \{-1, 0+1\}$$

the approximate angle $\tilde{\beta}$ is obtained by truncating the representation

$$\tilde{\beta} = \sum_{k=0}^{\tilde{w}} \sigma_k \arctan 2^{-k}\,, \text{ where } \tilde{w} < w\,.$$

Then, $\tilde{\beta}$ is implemented by realizing only the basis angles for which $\sigma_k \neq 0$. Let r be the number of $\sigma_k \neq 0$ for $k \in \{0, 1, \ldots, \tilde{w}\}$.

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The orthogonal Daubechies' filters can be implemented by a lattice filter using $N/2$ orthonormal rotations. The rotation angles β_i ($i = 1, \ldots, N/2$) are determined by the factorization of the scattering matrix

$$S(z) = \begin{bmatrix} H(z) & G(z) \\ H(-z) & G(-z) \end{bmatrix}$$

into a shift product of orthonormal rotations [6]. Note that for long filter lengths using the scattering matrix instead of the alias component matrix [18] yields better numerical results.

The efficient implementation of Daubechies' filter is obtained by implementing the orthogonal rotations of the lattice filters using approximate CORDIC–based μ–rotations [12, 13]. Given an accuracy $\Delta\beta_i$ for the approximate rotation angle $\tilde{\beta}_i$ (i.e. $\beta_i = \tilde{\beta}_i + \Delta\beta_i$) the approximate rotation angles are determined as described above. Note, that in many cases small values of r already yield reasonably accurate approximate wavelet transforms (e.g. see the approximate implementation of Daubechies' length $N = 4$ wavelet using $r = 1$ for all angles β_i [12]).

Table 1 and 2 and figure 1 and 2 show the results of this approximation procedure for Daubechies' wavelet of length $N = 10$. In table 1 the required angles of the original Daubechies' filter D_{10} and the angles of the approximate version $\tilde{D}_{10}$ are shown. The approximate version $\tilde{D}_{10}$ can be implemented requiring only 26 shift–add operations (see figure 1) while the standard lattice filter implementation of D_{10} requires 11 multiplications. The scaling function of this approximate realization is shown in figure 2 (dashed line). Remarkable is the fact, that the approximate scaling function is very close to the original scaling function (dotted line in figure 2) despite the significant reduction of the computational expense. The orthonormality is preserved exactly by the approximation procedure but the moments of the approximate wavelet do no longer vanish (only the zeroth moment is still vanishing since $\sum \tilde{\beta}_i = -45°$ can be guaranteed by the approximation procedure [6, 12]). The moments of the original and the approximate version are shown in table 2 and the respective zeros of $H(z)$ and $\tilde{H}(z)$ in figure 2. Note, that the value of the non-vanishing moments (distortion of the zeros of $H(z)$, respectively) is a function of the accuracy of the approximation of the rotation angles β_i, i.e. the complexity of the implementation (number of CORDIC–based μ–rotations used to approximate the exact angles).

	β_1	β_2	β_3	β_4	β_5
D_{10}	$-75.15°$	$46.77°$	$-23.02°$	$7.59°$	$-1.19°$
$\tilde{D}_{10}$	$-75.07°$	$46.79°$	$-22.95°$	$7.13°$	$-0.90°$

Table 1. Comparison of the angles of Daubechies' wavelet ($\mathbf{D_{10}}$) and its approximate version ($\tilde{\mathbf{D}}_{10}$).

3. APPROXIMATE MOMENTS AND REGULARITY

It was shown in [14, 15] that regular discrete wavelets and scaling sequences converge towards continuous functions as the number of scales J goes to infinity. If the actual wavelet transformation is only used up to a finite scale $j \to J < \infty$ (as it is usually the case in many signal processing discrete time applications) the limit functions for $j = J$ are discrete functions. We use the numerical derivatives of these discrete time versions of the scaling functions and their evolution for increasing J to analyse the effects of the approximate moments on the regularity of the scaling (wavelet) function. Given the scaling coefficients h_n ($n \in \{0, 1, \ldots, N-1\}$) we obtain the finite scale discrete scaling functions h_n^j ($n \in \{0, 1, \ldots, 2^j N - 1\}$) by computing

$$h_n^{j+1} = \sum_k h_k^j h_{n-2k} ,$$

where $h_n^0 = h_n$.

In figure 3 the scaling function and the first and second numerical derivative of the original Daubechies scaling function (plots in the left column) and the approximate scaling function (plots in the right column) are shown for scale $J = 7$. Obviously, the regularity ("smoothness") of the approximate scaling function is degraded, but the degrdation is small, since the moments are kept small by using an accurate approximation.

In many discrete applications where we only look at a certain number of scales J the convergence of the limit function (i.e. $j \to J$) is not an essential messure as in the continuous case ($j \to \infty$). As long as the scaling function and its respective *finite scale regularity* (regularity evaluated only up to a certain finite scale J) only show a boundeded variation from the original version, the approximate discrete wavelet transform will be as well suited as the exact discrete wavelet transform. In figure 4 the first numerical derivative with respect to scale is shown for scales $j = 5, 6, 7, 8$. Obviously, the finite scale regularity of the approximate version is almost as good as the regularity of the original version for the respective scales although in contrary to the original Daubechies' function the approximate function (divergent binary subdivision scheme for $j \to \infty$) is not differentiable.

Finally, a remark concerning this finite scale regularity of classical filter banks is worthwhile. For discrete applications only the finite scale regularity, i.e. the regularity meassure up to the actually used scale, is the essential meassure (what's happening between the sampling points is not of interest for many discrete applications). This explains the wide application and good results obtained by the filter bank community before wavelets, vanishing moments and smoothness were an issue. For example, the 8-tap scaling filter designed by Smith and Barnwell [17] diverges for $j \to \infty$. Therfore, it does not yield a smooth function. However, the obtained sequences (discrete functions) are finite scale regular for the actually used small J's. This is due to the fact that the scaling filter's transfer

moment	0	1	2	3	4
D_{10}	0.000	0.000	0.000	0.000	0.000
$\tilde{D}_{10}$	0.000	-0.001	0.078	1.236	13.660

Table 2. Comparison of the moments of $\mathbf{D_{10}}$ and $\tilde{\mathbf{D}}_{10}$.

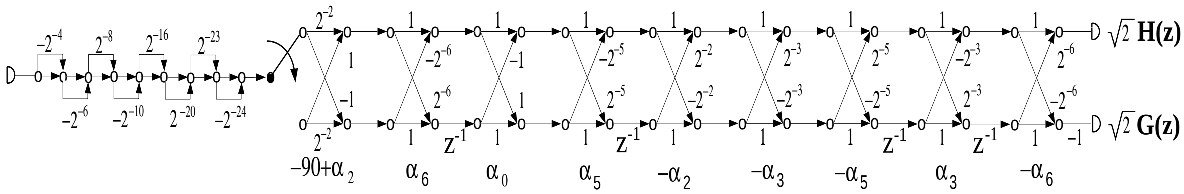

Figure 1. Implementation of approximate filter $\tilde{\mathbf{D}}_{10}$.

function is strongly attenuated in the stop band. Figure 5 shows the first numerical derivative for the Smith/Barnwell (SB) filter for different scales ($j = 3, 4, 5, 6$). As for the approximate version of Daubechies' filter (right column of figure 4) the first numerical derivative shows a bounded variation indicating a certain degree of "discrete smoothness". Obviously, the bound on the variation of the SB filter bank is much larger than for the approximate version of Daubechies' filter. Note particularly, that the variation of the derivative already occurs at lower scales for the SB filter than for $\tilde{D}_{10}$. Therefore, having in mind the excellent results of the classical filter banks, using approximate moments opens a wide range of possibilities not only from the implementation point of view but also for the design of wavelet transforms with approximately vanishing moments [11].

4. CONCLUSION

In this paper the approximate moments and the respective finite scale regularity of efficiently implemented orthonormal wavelet transforms was discussed. For many practical applications only the regularity up to the actually used finite scale is essential. For these finite scales the discrete functions (as well as their numerical derivatives) only exhibit a bounded variation from the original functions. Therefore, for many practical applications these wavelets are as well suited as the standard orthonormal wavelets, while the complexity of their implementation is significantly reduced. An architecture for the approximate versions is presented in [16].

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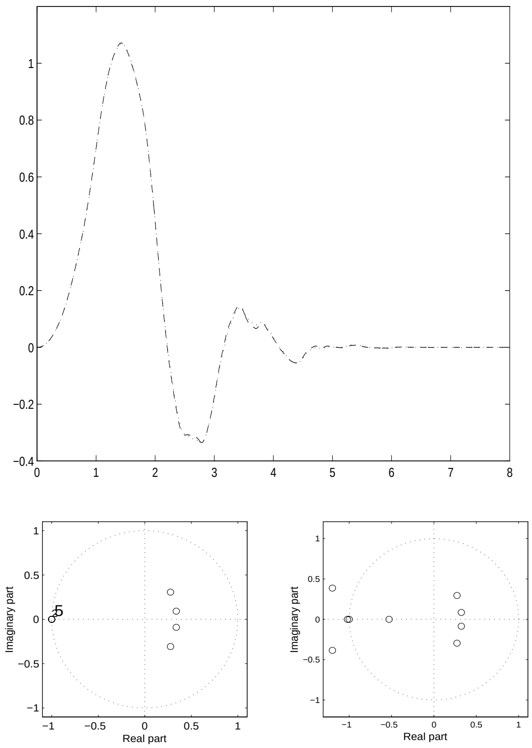

Figure 2. Scaling function of $\mathbf{D}_{10}$ (dotted line) and $\tilde{\mathbf{D}}_{10}$ (solid line) as well as the respective zero distribution of $\mathbf{H}(z)$ (left) and $\tilde{\mathbf{H}}(z)$ (right).

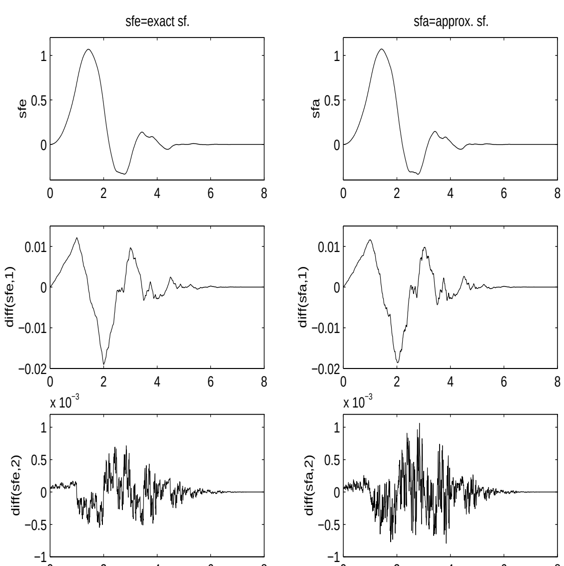

Figure 3. Scaling function and 1st and 2nd numerical derivative for $\mathbf{D}_{10}$ (left column) and $\tilde{\mathbf{D}}_{10}$ (right column) using $\mathbf{J = 7}$.

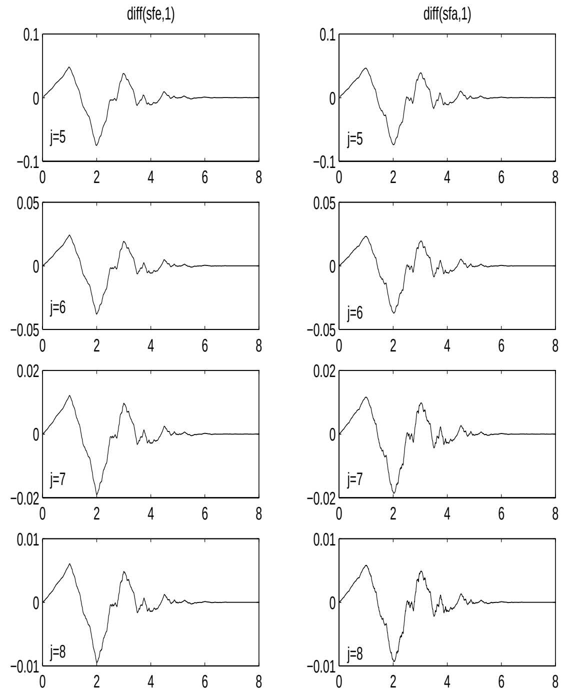

Figure 4. First numerical derivative with respect to scale ($\mathbf{J = 5, 6, 7, 8}$) for $\mathbf{D}_{10}$ (left column) and $\tilde{\mathbf{D}}_{10}$ (right column).

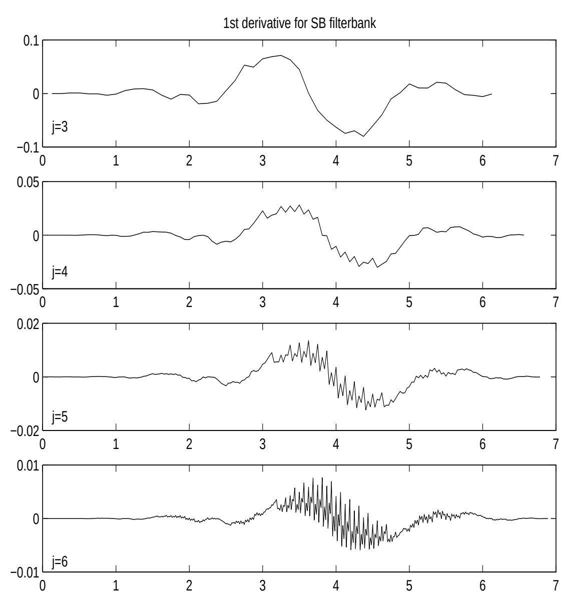

Figure 5. First numerical derivative with respect to scale ($\mathbf{J = 3, 4, 5, 6}$) for the Smith/Barnwell filter bank.