PARAMETERIZATION AND IMPLEMENTATION OF ORTHOGONAL WAVELET TRANSFORMS

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ABSTRACT

In this paper a method is presented that parameterizes orthogonal wavelet transforms with respect to their properties (i.e. compact support, vanishing moments, regularity, symmetry) and also takes into considerations a simple implementation of the transform. The parameter space is given by the rotation angles of the orthogonal 2×2 rotations used in the lattice filters that realize the different stages of the wavelet transform. The different properties of an orthogonal wavelet transform can be expressed in this parameter space. Then, restricting the parameter space to the rotation angles of simple CORDIC–based approximate rotations leads to a reduced parameter space. The wavelet transforms in this reduced parameter space are amenable to a very simple implementation (only a small number of shift & add operations).

I. INTRODUCTION

In recent years wavelet transforms have gained a lot of interest in many application fields, whereby orthogonal wavelet bases have been introduced by examining different possibilities for their design [1, 6]. Methods were presented to parameterize orthogonal wavelet transforms with respect to their properties, i.e. compact support, vanishing moments, regularity, and symmetry [3]. While most attention is focused on orthonormal, compactly supported wavelets with a maximal number of vanishing moments, also smoother, more regular wavelets, as well as least assymmetric, orthogonal wavelets [1] were designed.

Orthogonal lattice filters [9] can be used to implement the stages of orthogonal wavelet transforms within a filterbank structure. Therefore, each filter consists of orthogonal 2×2–rotations. The angles of these rotations represent our parameter space.

The CORDIC algorithm [10] offers one possibility to execute elementary rotations, whereby a sequence of μ–rotations being implementable with a few shift- and add-operations is used. CORDIC-based approximate rotations [5, 4, 7] renounce on the full sequence of μ–rotations by using only a few μ–rotations such that the computational complexity is significantly reduced. Note, however, that the orthogonality is preserved independent of the approximation accuracy.

In this paper the design of orthogonal wavelet transforms with certain properties is described in the parameter space (i.e.the rotation angles of the lattice filters). Then, the full parameter space with all possible rotation angles is reduced to the angles that can be implemented by only one CORDIC–based μ–rotation (i.e. one pair of shift & add operations). Therefore, the simple implementation (only a few shift and add operations) of the designed wavelet transform is guaranteed. The prize one has to pay, namely the reduced amount of solutions does not affect the quality of the transforms with respect to many practical applications.

This paper is organized as follows: In section 2 the parameterization of orthogonal wavelet transforms is discussed and typical properties of wavelet bases are reviewed. Section 3 deals with CORDIC–based approximate rotations that allow the efficient implementation of elementary 2×2 rotations. In section 4 the restriction of the full parameter space to the CORDIC–based μ–rotation angles is outlined resulting in wavelet transforms that are amenable to a simple implementation.

II. PARAMETERIZATION OF ORTHOGONAL WAVELET TRANSFORMS

An orthogonal wavelet transform decomposes a signal in dilated and translated versions of the wavelet function $\Psi(t)$. The wavelet function $\Psi(t)$ is based on a scaling function $\Phi(t)$ and both can be represented by

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dilated and translated versions of this scaling function.

$$\Phi\left(t\right) = \sum_{i=0}^{n-1} h_i \Phi\left(2t-i\right) \qquad \Psi\left(t\right) = \sum_{i=0}^{n-1} g_i \Phi\left(2t-i\right)$$

With these coefficients h_i and g_i the transfer functions of the digital filters that are used to implement the discrete orthogonal wavelet transform can be formulated.

$$H(z) = \sum_{i=0}^{n-1} h_i z^{-i}, \qquad G(z) = \sum_{i=0}^{n-1} g_i z^{-i}$$

A suitable architecture for the implementation of orthogonal wavelet transforms are lattice filters. Figure 1 shows a lattice filter implementation of one stage of Daubechies' wavelet transform of length $n = 4$, which is denoted by O_4^D. Obviously, the basic modules of the filter are orthogonal 2×2 rotations. By using these orthogonal rotations the orthogonality of the whole transform is structurally imposed [9]. For the lattice filter to perform an orthogonal wavelet transform, another property is necessary. This property ensures that the wavelet function is zero mean, what is equivalent with the wavelet having at least one vanishing moment and the transfer functions $G(z)$ and $H(z)$ having at least one zero at $z = 1$ and $z = -1$, respectively. In [11] it was shown, that these conditions are fullfilled if the sum of all rotation angles β_k is constant:

$$\sum_k \beta_k = -45^o \, .$$

Therefore, a lattice filter whose sum of all rotation angles is -45^o performs an orthogonal wavelet transform, independent of the angles β_k. As an example, a wavelet transform of length $n = 6$ is used that consists of 3 orthogonal rotations with angles β_1, β_2 and β_3. By choosing

$$\beta_1 = -45^o - \gamma \qquad \beta_2 = \gamma + \delta \qquad \beta_3 = -\delta \qquad (1)$$

all wavelet transforms of length $n = 6$ can be parameterized by varying γ and δ. Of course, not all pairs of (γ,δ) lead to a suitable wavelet transform. How the parameters (γ,δ) must be chosen depends on the desired properties. In the following, these properties are discussed:

Compact Support: The compact support is equivalent to the length n of the wavelet basis. While with $\gamma \neq 0$ and $\delta \neq 0$ wavelets with support $n = 6$ can be constructed, by setting δ to zero n is reduced to $n = 4$. In Figure 3 (upper left, dotted line) Daubechies' scaling function of compact support $n = 4$ (version O_4^D) is shown, whereby $\gamma = 15^o$.

Vanishing Moments: p vanishing moments are equivalent to the polynomial $G(z)$ having p zeros at $z = 1$ and the coefficients g_i fulfilling p of the following equations:

$$\sum_i i^j g_i = 0 \qquad j = 0, \ldots p-1. \qquad (2)$$

In Figure 3 (upper right, dotted line) the Daubechies' scaling function of length $n = 6$ is plotted. O_6^D shows $p = 3$ vanishing moments ($\gamma = 22.60^o$, $\delta = 6.03^o$).

Regularity: A wide–spread method to compute the Hölder regularity r numerically was given in [8]. The polynomial $H(z)$ representing the scaling function is analyzed.

$$H(z) = (1 + z^{-1})^p F_p(z)$$

implies that $H(z)$ has p vanishing moments. With $F_p(z) = f_{p,0} + f_{p,1} z + f_{p,2} z^2 + \ldots + f_{p,n-p-1} z^{n-p-1}$ the matrix $\boldsymbol{F}_p$ is given by

$$\boldsymbol{F}_p = \begin{bmatrix} f_{p,1} & f_{p,0} & 0 & \ldots & 0 \\ f_{p,3} & f_{p,2} & f_{p,1} & & \\ f_{p,5} & f_{p,4} & f_{p,3} & & \vdots \\ \vdots & & & \ddots & \\ 0 & & \ldots & & f_{p,n-p-2} \end{bmatrix}.$$

An upper bound r for the Hölder coefficient is

$$r = p - 1 - \log_2 \max(\mid f_{p,0} \mid, \mid f_{p,n-p} \mid, \rho(\boldsymbol{F}_p)),$$

where $\rho(\boldsymbol{F}_p) = \max_i(\mid \lambda_i \mid)$ is the spectral radius of $\boldsymbol{F}_p$ (λ_i are the eigenvalues of $\boldsymbol{F}_p$).

In Figure 3 (lower right, dotted line) the most regular scaling function of length $n = 6$ (O_6^R) is plotted ($\gamma = 26.06^o$, $\delta = 8.40^o$).

Symmetry: As symmetry is a prefered property for image coding, least asymmetric wavelets were designed for this wide–spread wavelet application. In [1] Coiflets are discussed, that show an improved symmetry in comparison to standard wavelets. These Coiflets can be designed, if not only the wavelets have vanishing moments, but also the scaling functions. The scaling function of version O_6^S with $n = 6$ and $p = 2$ ($\gamma = -122.85^o$, $\delta = -192.15^o$) is plotted in Figure 3 (lower left, dotted line).

III. EFFICIENT IMPLEMENTATION OF ORTHOGONAL ROTATIONS

An orthogonal 2×2–rotation $\boldsymbol{R}(\alpha)$ is defined as follows:

$$\boldsymbol{R}(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

The CORDIC algorithm is a common method to execute orthogonal rotations by using a sequence of $w+1$ μ–rotations (w being the wordlength):

$$\boldsymbol{R}(\alpha) = \frac{1}{K_w} \prod_{k=0}^{w} \begin{bmatrix} 1 & -\sigma_k 2^{-k} \\ \sigma_k 2^{-k} & 1 \end{bmatrix}, \quad \sigma_k \in \{+1, -1\},$$

with $\frac{1}{K_w} = \prod_{k=0}^{w} \frac{1}{\sqrt{1+2^{-2k}}}$ being the scaling factor. This corresponds to the representation of α as

$$\alpha = \sum_k \sigma_k \alpha_k = \sum_k \sigma_k \mathrm{arctan} 2^{-k}.$$

This representation of an angle in the ”$\arctan 2^{-k}$” basis is also the basic idea of CORDIC–based approximate rotations [5, 4], but there we have $\sigma_k \in \{-1, 0, +1\}$.

In [5] double rotations consisting of 2 equal CORDIC μ–rotations were used, which rotate by the angle $2\alpha_k$, i.e. $\boldsymbol{R}(2\alpha_k) = \boldsymbol{R}(\alpha_k)\boldsymbol{R}(\alpha_k)$ such that

$$\boldsymbol{R}(2\alpha_k) = \frac{1}{K_k^2} \begin{bmatrix} 1 & -\sigma 2^{-k} \\ \sigma 2^{-k} & 1 \end{bmatrix} \begin{bmatrix} 1 & -\sigma 2^{-k} \\ \sigma 2^{-k} & 1 \end{bmatrix}$$

Now, the scaling factor $\frac{1}{K_k^2}$ can be factorized into a sequence of shift & add operations [2]:

$$\frac{1}{K_k^2} = \frac{1}{1+2^{-2k}} = (1-2^{-2k})(1+2^{-4k})(1+2^{-8k})\ldots \tag{3}$$

With one (or a few) of these double rotations an approximate rotation can be composed, that is simple to implement, approximates any rotation angle to a certain accuracy (increasing the number of double rotations increases the accuracy), and is always exactly orthogonal independent of the accuracy.

IV. RESTRICTION OF THE PARAMETER SPACE

For the parameterization of wavelet transforms, two items of section 2 are important. The orthogonality of the transforms is structurally imposed by the lattice structure. By choosing the rotations such that the sum of angles is constant -45^o, the lattice structure always performs a wavelet transform. Therefore, the rotation $\boldsymbol{R}(-45^o)$ always appears once in the presented wavelet filters:

$$\boldsymbol{R}(-45^o) = \boldsymbol{R}(-\alpha_0) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

The scaling factor $1/\sqrt{2}$ is not critical for the implementation, since it is not necessary to realize it. With each rotation appearing in the analysis part and in the synthesis part, also the scaling factor appears twice. As $1/\sqrt{2} \cdot 1/\sqrt{2} = 1/2$ can be implemented with one shift operation, the only price that must be paid is the loss of normality by a factor $1/\sqrt{2}$ in the transform domain.

By using only one (instead of $w+1$) μ–rotation per parameter γ and δ of (1), i.e.

$$\gamma = \pm\mathrm{arctan} 2^{-k_\gamma} \qquad \delta = \pm\mathrm{arctan} 2^{-k_\delta}$$

the computational complexity is reduced drastically. These μ–rotations appear twice in our parameterization schemes (see e.g. (1)). Therefore, it is always a double rotation (with different signs) that must be implemented. This also implies that the simple realization of the scaling factor (3) can be applied.

The price one has to pay for the simplicity of the filters is, that the parameter space consisting of all possible rotation angles is reduced to a discrete parameter space spanned by the CORDIC-angles $\alpha_k = \arctan 2^{-k}$. But by using different parameterization schemes -not only (1)- that all show a constant sum of angles $\sum_k \beta_k = -45^o$, the grid of the reduced parameter space is becoming more dense. Of course, it is also possible to increase the accuracy by using more than one μ–rotation per parameter.

The decisive question is whether or not the reduced parameter space leads to suitable wavelet transforms with respect to the discussed properties, namely vanishing moments, regularity and symmetry. In Figure 3 the wavelets, parameterized in the full parameter space O_4^D, O_6^D, O_6^R, O_6^S (dotted lines) are compared to the versions $\tilde{O}_4^D$, $\tilde{O}_6^D$, $\tilde{O}_6^R$, $\tilde{O}_6^S$ (solid lines) designed in the reduced parameter space. Table 1 compares the amount of vanishing moments p, the upper bound of the Hölder regularity r and the used rotation angles β_k of the different versions. Obviously, the simple versions approximate the standard versions very well although their implementation is remarkably simple compared to the standard versions. As an example the implementation of $\tilde{O}_4^D$ is shown in Figure 2. The amount of necessary operations is 7 shift & add operations (w=32). The reduced computational expense is based on the fact that only one μ–rotation is used per parameter.

V. CONCLUSION

In this paper a method was presented to parameterize orthogonal wavelet transforms with respect to certain properties. Besides the standard properties (i.e. compact support, vanishing moments, regularity, symmetry) a simple implementation of the wavelet transforms is taken into consideration. Using only one very simple CORDIC–based μ–rotation per parameter (rotation angle of the lattice filter) guarantees the very simple implementation of the transform. Different parameterization schemes, different types of μ–rotations as well as more than only one μ–rotation can be used to design wavelets which are closer to the standard versions. However, the most simple approach (one μ–rotation per parameter) already leads to wavelets which approximate the optimal versions very well, such that for many practical applications these fast/simple wavelet transforms perform as good as the standard versions.

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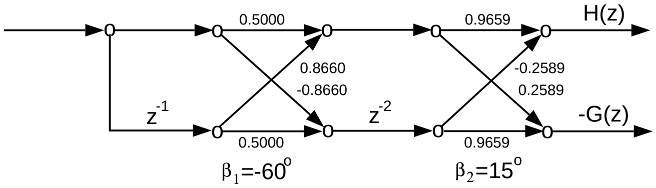

Figure 1: Lattice filter for implementing O_4^D.

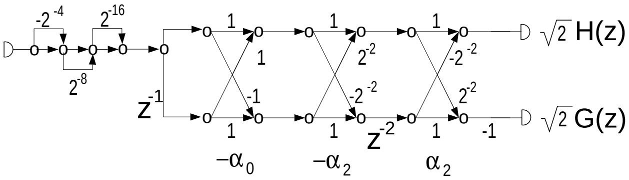

Figure 2: Lattice filter for implementing $\tilde{O}_4^D$.

Version	p	r	β_1	β_2	β_3
O_4^D	2	0.5500	-60.00^o	15.00^o	0
$\tilde{O}_4^D$	1	0.5025	-59.04^o	14.04^o	0
O_6^D	3	1.0878	-67.60^o	28.63^o	-6.03^o
$\tilde{O}_6^D$	1	1.0597	-68.84^o	30.96^o	-7.12^o
O_6^R	2	1.4176	-71.06^o	34.46^o	-8.40^o
$\tilde{O}_6^R$	1	1.0612	-71.56^o	33.69^o	-7.12^o
O_6^S	2	0.5424	77.85^o	69.30^o	-192.15^o
$\tilde{O}_6^S$	1	0.5411	75.96^o	66.16^o	-187.12^o

Table 1: Comparison of the different versions of orthogonal wavelet transforms

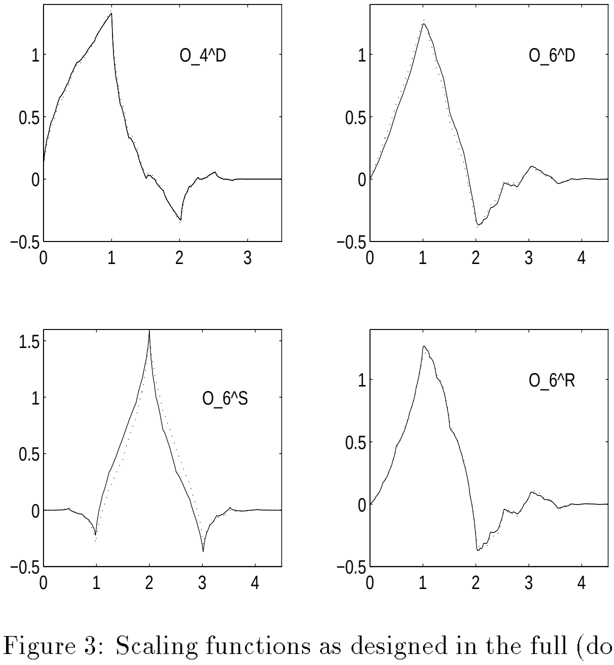

Figure 3: Scaling functions as designed in the full (dotted lines) and in the reduced (solid lines) parameter space with one μ–rotation per parameter.