EXIT-Optimized Index Assignments for Turbo Decoders with Unreliable LLR Transfer

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***Abstract*—In this letter a Turbo decoder is considered, for which the log likelihood ratios (LLRs) exchanged between constituent decoders are subject to independent bit errors. The resilience of the decoder against these errors can be improved - without increasing decoding complexity - by optimizing the index assignment (IA) of the LLRs with respect to the extrinsic information transfer (EXIT) characteristic of the decoder. To enable an effective optimization of the IA we propose an approximation of the mutual information of the extrinsic decoder output that can be computed efficiently without histogram measurements. Numerical results in terms of bit error rates are reported for different scenarios to validate the proposed approach.**

I. Introduction

The design of signal processing algorithms which are resilient against unreliable underlying circuits has gained increasing interest lately. On the one hand, increased integration density makes integrated circuits (ICs) more vulnerable to soft error events originating from process variations or harsh operating conditions [1]. On the other hand, hardware faults may be introduced deliberately if techniques like aggressive voltage scaling are employed to improve energy efficiency of an IC [2], [3]. A few authors have recently considered the required co-design on algorithmic and circuit level for channel decoders, which are among the most complex, integral parts in baseband processing: It has been shown that the resilience against corrupted log likelihood (LLR) values of Viterbi decoding [4], iterative decoding [5] and Turbo equalization [6], can be improved if the decoder employs a suitable stochastic model of the joint hardware and communication channel distortions. For the case of LDPC decoding, the authors of [7] propose to protect the most significant bits of the underlying LLRs using additional redundancy. In [8] it is suggested to improve the resilience of a Turbo decoder against LLR errors on algorithmic level by executing additional decoding iterations. A detailed analysis of the effect of unreliable LLR storage in the receive buffer of a channel decoder is provided in [9], where the authors emphasize the importance of a carefully designed index assignment of the quantizer output. The authors have recently extended their work in [10] and investigated application specific index assignments for the case of repetition coding and convolutional coding.

In this letter a Turbo decoder with unreliable transfer of extrinsic LLR values between the constituent decoders is considered. This transfer is modeled such that the binary representation of the LLRs is subject to a binary symmetric channel (BSC). It was already shown in [11], that the impact of this disturbance can be reduced if the transition metric of the log-domain BCJR algorithm is based on the transfer function of cascaded consistent Gaussian channel and BSC. In [11] we have also proposed an extrinsic information transfer (EXIT) based optimization of the LLR's index assignment. However, while yielding additional resilience against LLR disturbance, this also poses a complex combinatorial optimization problem, that typically has to be tackled using a meta-heuristic approach. In order to cope with the complexity of the problem's EXIT-based objective function, which has to be evaluated repeatedly during the optimization process, this letter describes an approximation of the EXIT function for arbitrary index assignments, that can be computed analytically. This enables an efficient optimization without time-consuming histogram measurements of the EXIT function. It is shown that the resulting optimized index assignments significantly improve the bit error rate (BER) performance of the Turbo decoder under unreliable LLR transfer without increasing the computational complexity of the decoder.

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II. System Model

The transmission of BPSK-modulated Turbo coded data with rate R over a channel is modeled as shown in Fig. 1: Binary information symbols u are encoded using two parallel concatenated recursive systematic convolutional codes, such that the coded symbols v consist of a systematic part and two parity parts from first and second encoder. BPSK modulation of v results in the equally probable modulation symbols $x = 2v - 1$. The communication channel is modeled as an independent Rayleigh fading channel or additive white Gaussian noise (AWGN), such that the received values r can be written as

$$r = ax + n, \qquad \text{where } n \sim \mathcal{N}(0, \sigma_r^2), \tag{1}$$

where σ_r^2 denotes the noise power of the channel and a is Rayleigh distributed with $E\{a^2\} = 1$ or equal to one for Rayleigh or AWGN channel, respectively. The channel is characterized by its signal-to-noise ratio $E_b/N_0 = -10\log_{10} 2R\sigma_r^2$. Based on the received symbols r the Turbo decoder computes an estimate $\hat{u}$ of the original information by iterating between two constituent decoders: To produce extrinsic LLRs λ_E of the information symbols, the first decoder employs the systematic part r_s and the first parity part r_1 of r, while the second decoder uses the interleaved systematic part r_s' and the second parity part r_2 of r. The interleaved/de-interleaved LLRs λ generated by the first/second decoder become the *a priori* LLRs $\overline{\lambda}$ of the second/first decoder.

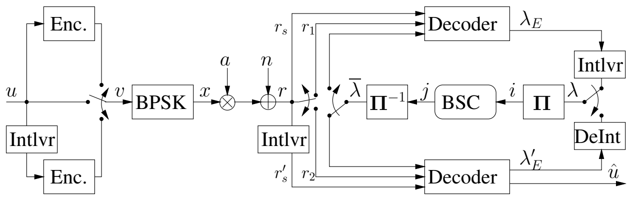

Fig. 1. Turbo coded transmission over Rayleigh/AWGN channel with unreliable LLR transfer between the constituent decoders.

In a decoder implementation both r and λ are subject to quantization. Being only concerned with the effects of the unreliable exchange of λ, we assume r to be of sufficient accuracy and do not consider it in the following. The LLR values are elements of a set of reconstruction values $\mathcal{Q}$ using a two's complement quantization scheme:

$$\lambda \in \mathcal{Q}, \text{ where } \mathcal{Q} = \{-2^{d-1} + k2^{-f} : 0 \leq k < 2^N\},$$

where $N = d + f$ and d and f denote the number of decimal and fractional bits. In a conventional system the binary representation of $k \oplus 2^{N-1}$ is written to a buffer, from where the other decoder restores the LLRs in the next decoding stage. Under the assumption that this LLR transfer is unreliable, for instance because of an error-prone buffer memory, the restored LLRs $\overline{\lambda}$ may differ from the original LLRs λ. This distortion is modeled as a BSC [9], [12] with input and output a binary representation of λ and $\overline{\lambda}$, respectively, and bit error probability p_e.

Instead of employing $i = k \oplus 2^{N-1}$, any other mapping from quantization values to indices i may be used. This *index assignment* (IA) can be defined by a bijective function $i = \Pi(k)$. The choice of the IA Π influences the distortion of the reconstructed values for $p_e > 0$, while it is not relevant for $p_e = 0$. Given N quantization bits there exist $(2^N)!$ possible IAs, where for each IA there exist $(2^N - 1)N!$ equivalent IAs [13]. A special case is natural binary coding (NBC), where $i = k$, and the equivalent two's complement representation, $i = k \oplus 2^{N-1}$, which are known to be optimal in mean squared error (MSE) sense for uniform, equiprobable symbols over the BSC [14]. In general the selection of an IA subject to an objective function is, however, a complex combinatorial optimization problem. The choice of the objective function should depend on the application and a conventional choice is naturally the MSE criterion (e.g. [15]), where the average distortion of the reconstructed quantization values is to be minimized.

In context of Turbo decoding, we have to find an IA for LLR values with statistical properties, that vary depending on the current iteration progress of the decoder. For implementation reasons it is not advisable to determine different IAs for different decoding situations, but to find a single IA, that provides optimal protection in situations where the iteration progress is most sensitive to LLR errors. To this end, an EXIT chart based design criterion can be employed, which is detailed in the following Sec. III-A.

III. EXIT Optimized Index Assignments

A. Optimization Problem

The EXIT chart [16] of the decoder with unreliable LLR transfer is determined using the model shown in Fig. 2. The decoder employs the two inputs r and $\overline{\lambda}$ to generate an extrinsic estimate λ_E of u. The output of the communication channel is r, which results from the convolutionally encoded and BPSK modulated transmission of u over an AWGN channel. For the *a priori* channel u is BPSK modulated yielding x_s, scaled with $\sigma_\lambda^2/2$ and subject to AWGN with σ_λ^2, such that the input $\tilde{\lambda}$ of the quantizer is consistently Gaussian distributed. The conditional probability mass function (PMF) of the quantizer output can thus be calculated for each $\lambda \in \mathcal{Q}$ as

$$P(\lambda|x_s) = \frac{1}{\sqrt{2\pi}\sigma_\lambda} \times \begin{cases} \int_{-\infty}^{\lambda+\delta} e^{-\left((\tilde{\lambda} - x_s\sigma_\lambda^2/2)^2/(2\sigma_\lambda^2)\right)} d\tilde{\lambda}, & \lambda = -2^{d-1} \\ \int_{\lambda-\delta}^{\infty} e^{-\left((\tilde{\lambda} - x_s\sigma_\lambda^2/2)^2/(2\sigma_\lambda^2)\right)} d\tilde{\lambda}, & \lambda = 2^{d-1} - 2^{-f} \\ \int_{\lambda-\delta}^{\lambda+\delta} e^{-\left((\tilde{\lambda} - x_s\sigma_\lambda^2/2)^2/(2\sigma_\lambda^2)\right)} d\tilde{\lambda}, & \text{else,} \end{cases}$$

where $\delta = 2^{-f-1}$. Also, it is easy to see that the PMF with respect to the indices k is given as

$$P(k|x_s) = P(\lambda = -2^{d-1} + k2^{-f}|x_s).$$

Consequently, the average mutual information (MI) at quantizer output $I(U;\Lambda)$ forms the *a priori* MI of the decoder for error-free LLR transfer. It can be computed [17] as

$$I(U;\Lambda) = \frac{1}{2} \sum_{x_s=\pm 1} \sum_{k=0}^{|\mathcal{Q}|-1} P(k|x_s) \log_2 \frac{P(k|x_s)}{P(k)}.$$

If we model the unreliable LLR transfer as shown in Fig. 2 by transmitting a binary representation of λ over a successive BSC, the actual MI $I(U;\overline{\Lambda})$ at decoder input is degraded by

$$\Delta I(p_e, \Pi) = I(U;\Lambda) - I(U;\overline{\Lambda}) > 0 \quad \text{for} \quad p_e > 0. \qquad (2)$$

Note that (2) directly follows from the data processing theorem, because the *a priori* channel can be seen as a Markov chain $U \rightarrow \Lambda \rightarrow \overline{\Lambda}$ for which it holds that $I(U;\Lambda|\overline{\Lambda}) > 0$ for $p_e > 0$ and $I(U;\overline{\Lambda}|\Lambda) = 0$.

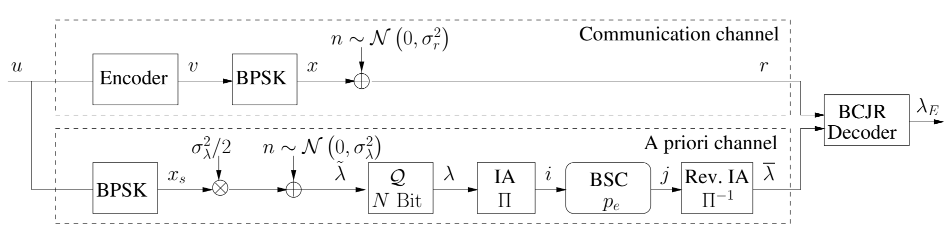

Fig. 2. Determination of the EXIT chart for a Turbo decoder with unreliable LLR transfer.

For fixed $\mathcal{Q}$ and σ_λ^2, the MI loss $\Delta I(p_e, \Pi)$ depends on p_e and on Π. The actual MI $I(U; \overline{\Lambda})$ at decoder input is based on the PMF $P(\overline{\lambda}|x_s) = P(\Pi^{-1}(j)|x_s)$ with

$$P(j|x_s) = \sum_{i=0}^{|\mathcal{Q}|-1} P(i = \Pi(k)|x_s)(1-p_e)^{N-d_H(i,j)} p_e^{d_H(i,j)}. \quad (3)$$

In (3) $d_H(i,j)$ is the Hamming distance between i and j.

The EXIT chart of the decoder visualizes the decoding behavior by plotting the *a priori* MI $I(U;\Lambda)$ and the extrinsic MI $I(U;\Lambda_E)$ of both constituent decoders against each other. Convergence is given if the "EXIT tunnel" is open for all $I(U;\Lambda)$, i.e. for Turbo codes with two identical encoders if $I(U;\Lambda_E) > I(U;\Lambda)$. It follows that for a given E_b/N_0, p_e and $\mathcal{Q}$, an optimization of

$$\max_{\Pi} \min_{\sigma_\lambda^2} \{I(U;\Lambda_E) - I(U;\Lambda)\} \quad (4)$$

yields an IA, that provides highest resilience where the EXIT curves are closest and thus the decoding process is most sensitive. Employing such an IA can therefore improve decoding performance without impact on decoding complexity. However, optimizing (4) is a complex, combinatorial optimization problem. Metaheuristics have been proposed to solve this type of problem, for example simulated annealing [18].

B. Approximated Computation of Objective Function

An optimization of (4) involves the repeated computation of $I(U;\Lambda_E) - I(U;\Lambda)$ for the candidate IAs, where each computation involves measuring $I(U;\Lambda_E)$ for a suitable number of different σ_λ^2. This is usually done based on histogram measurements, which require a sufficiently high block length to get a meaningful estimate of $I(U;\Lambda_E)$. The resulting time consuming process will impair an effective optimization.

Alternatively we propose to use an approximation of $I(U;\Lambda_E)$ that can be computed analytically based on a previously determined $I(U;\Lambda_E)$ for error-free LLR transfer: For a given E_b/N_0, error probability p_e and quantizer $\mathcal{Q}$ let

- $I_A(\sigma_\lambda^2)$ denote the *a priori* MI $I(U;\Lambda)$ as a monotonically increasing function of the noise variance σ_λ^2, and
- $I_E(I_A(\sigma_\lambda^2), \Pi)$ denote the extrinsic MI $I(U;\Lambda_E)$ as a function of $I_A(\sigma_\lambda^2)$ and the IA Π.

Assume that the extrinsic MI $I_E^\star(I_A(\sigma_\lambda^2))$ for perfect LLR transfer has been determined beforehand. Note that $I_E^\star(I_A(\sigma_\lambda^2))$ is independent of Π, because the IA has no influence if $p_e = 0$. Then the extrinsic MI for an arbitrary IA Π and unreliable transfer may be approximated as

$$\hat{I}_E(I_A(\sigma_\lambda^2), \Pi) = I_E^\star(I_A(\sigma_\lambda^2) - \Delta I(p_e, \Pi)) \quad (5)$$

and the optimization is simplified to

$$\Pi^* = \arg\max_{\Pi} \min_{\sigma_\lambda^2} \left\{ \hat{I}_E(I_A(\sigma_\lambda^2), \Pi) - I_A(\sigma_\lambda^2) \right\}. \quad (6)$$

Thus the objective function in (6) can be efficiently evaluated for arbitrary Π by computing the MI loss $\Delta I(p_e, \Pi)$ using (2) and (3) and approximating the extrinsic MI based on (5), where $I_E^\star(I_A(\sigma_\lambda^2))$ has to be determined only once by suitable histogram measurement.

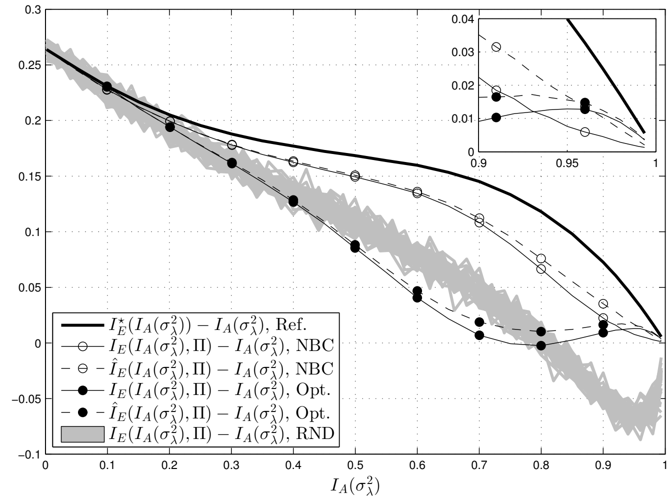

Fig. 3. Comparison of MI gain as function of *a priori* MI for different IAs and approximated and actual extrinsic MI ($p_e = 0.01$, $E_b/N_0 = 0.9dB$).

The approximation in (5) is more accurate if p_e is smaller: For increasing p_e the PMF $P(\overline{\lambda}|x_s)$ deviates from the original quantized bimodal Gaussian distribution $P(\lambda|x_s)$, as can be seen from (3). Thus even if $I(U;\Lambda)$ and $I(U;\overline{\Lambda})$ are equal, the shape of the underlying PMFs may differ significantly. As the extrinsic MI that the decoder produces is not just a function of the MI, but depends on the actual distribution of the input values, $I(U;\Lambda_E)$ will differ from the approximation in (5). However, it turns out that it is sufficiently accurate for the optimization procedure as will be shown in the following.

IV. Numerical Results

The following simulation results are obtained using a binary Turbo code with $R = 1/3$, generator $\mathbf{G}(D) = \left[1,\ \frac{1+D+D^3}{1+D^2+D^3}\right]$ and random interleaver of length 2^{15}. The decoder is based on the LogMAP algorithm with perfect channel state information. The transition metric incorporates the extrinsic information based on $\log P(\overline{\lambda}|x_s)$ [11]. The LLR values are quantized using $N = 5 + 2 = 7$ bits.

Simulated annealing [18] is applied to optimize (6) for different p_e, where an initial and final temperature of $T_0 = 2$ and $T_f = 10^{-12}$, respectively, and a cooling factor of $\alpha = 0.99$ are used. Optimization terminates early if $5 \cdot 10^4$ consecutive pertubations do not yield an improvement. As an example, Fig. 3 illustrates the result of the optimization for the case of $p_e = 0.01$ and $E_b/N_0 = 0.9dB$: It shows the increase of MI that the decoder produces for a given *a priori* MI $I_A(\sigma_\lambda^2)$. This may be referred to as the MI gain of the decoder and is actually the objective function of (6). For the decoder to converge, this function must be greater than zero before it reaches zero at $I_A(\sigma_\lambda^2) = 1$. The range just before $I_A(\sigma_\lambda^2) = 1$ is especially critical to obtain *low* BERs. It corresponds to the upper right corner of the conventional EXIT chart of a Turbo decoder as introduced in [16].

The MI gain for error-free LLR transfer is shown as reference ("Ref."). If LLRs are distorted, the performance will be degraded and the IA, which has no influence for $p_e = 0$, becomes important:

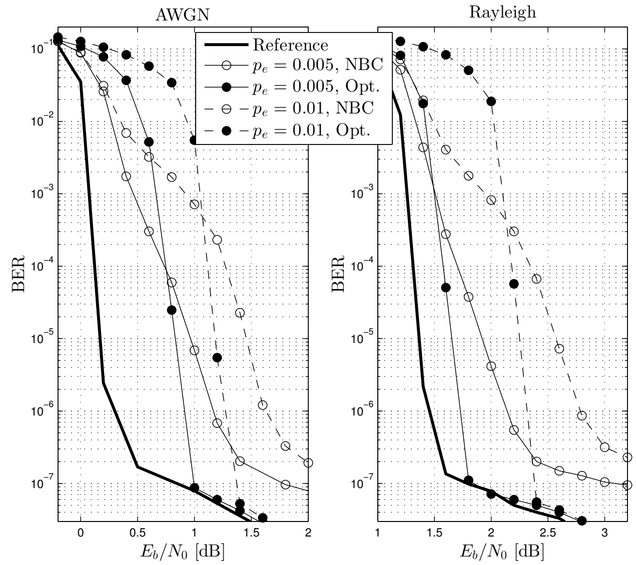

Fig. 4. BER after 8 decoder iterations for NBC IA and optimized IA and for different error probabilities p_e and AWGN and Rayleigh channels.

- The curve for random IA ("RND") has been produced by randomly permuting the IA at each simulated *a priori* MI value and for 100 realizations. It shall give an impression of the average performance of an arbitrarily selected IA. A strong performance degradation is visible, where the decoder is not able to improve the MI for $I_A(\sigma_\lambda^2) > 0.8$.
- If NBC is employed, performance is improved compared to random IA and convergence can be expected, because $I_E(I_A(\sigma_\lambda^2), \Pi) - I_A(\sigma_\lambda^2) > 0$ for all $I_A(\sigma_\lambda^2)$. It, however, approaches zero closely for $I_A(\sigma_\lambda^2) > 0.95$, which indicates a possible performance loss for lower BERs.
- In contrast, the IA ("Opt."), which results from optimizing (6), exhibits a significantly differently shaped MI gain function. While it performs worse than NBC IA for $I_A(\sigma_\lambda^2) < 0.94$, it is more robust in the remaining range, which indicates a slightly higher BER threshold but improved performance for low BERs.

Besides the measured MI, Fig. 3 also shows the approximated functions, that are used during the optimization process, as dashed curves. A good match of the shapes of proposed approximation and actual measured function can be observed.

For Rayleigh and AWGN channels the benefit of the optimized IA is investigated in terms of BER in Fig. 4 for $p_e = 0.005$ and $p_e = 0.01$. Again, the reference is the decoder with error-free, quantized LLR transfer. If the LLR transfer is unreliable, the decoding performance is degraded: For instance, assuming a target BER of 10^{-5} and NBC, a loss of coding gain of about $0.8dB$ and $1.3dB$ and of about $0.6dB$ and $1.3dB$ can be observed in AWGN and Rayleigh case, respectively. Additionally the decoder exhibits an increased error-floor. We have optimized the IA using (6) at $E_b/N_0 = 0.3dB$ ($p_e = 0.005$) and $E_b/N_0 = 1.3dB$ ($p_e = 0.01$). The resulting optimized IAs yield a performance improvement of about $0.1dB$ and $0.3dB$ for the AWGN channel and $0.3dB$ for the Rayleigh channel. Moreover, the decoder even approaches the error-floor of the original decoder with error-free LLR transfer.

V. Conclusion

As shown in earlier work [4]–[6], [11], resilience against unreliable LLR transfer can be improved by matching decoder metrics to the actual error distribution. Adequate design of the LLR's index assignment can provide additional resilience. Although the commonly adopted NBC is better than "randomly" assigned indices, an optimization of the IA based on the EXIT characteristic of the Turbo decoder yields yet improved performance without increasing computational complexity of the decoder itself. In this letter we have shown that an easily computable approximation of the objective function enables efficient optimization of this complex combinatorial problem. Supporting numerical results have been provided for AWGN and Rayleigh fading channels.

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