SIGNAL SUBSPACE METHODS FOR DELAY ESTIMATION IN WIRELESS CDMA SYSTEMS

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ABSTRACT

We consider the problem of delay estimation in an asynchronous wireless CDMA communication system. The received observations are decomposed into orthogonal signal and noise subspaces via the singular value decomposition (SVD). A class of techniques known as subspace fitting techniques is used to estimate the individual user delays by maximizing the fit between the signal subspace and the observed data. Since we only require knowledge of the signal subspace, the noise subspace can be *sphericalized*, i.e., reduced to one dimension, which leads to a reduction in computational complexity over the full SVD. A computationally efficient algorithm is used to update the signal subspace and spherical noise subspace for each newly received observation vector. The delays can then be estimated in closed form through the roots of a quadratic equation.

I. INTRODUCTION

Direct Sequence Code Division Multiple Access (DS-CDMA) systems have been designed in the context of wireless mobile communications. Due to the innate asynchronous nature of the system, the base station receives different users' transmissions at different delays relative to the local clock. To facilitate subsequent detection of the transmitted bits, these delays (or timings) have to be accurately estimated. The two step process of estimating and tracking the delays is called synchronization.

Most of the work done in timing acquisition in CDMA systems involves computationally intensive joint parameter estimation for all the users [1, 2]. In [3] an elegant and computationally efficient algorithm is presented that reduces the multidimensional problem to one dimension[1]. This method is based on an eigenvalue decomposition of the correlation matrix of the observations into two orthogonal subspaces, the *signal* subspace and the *noise* subspace (subspace-based parameter estimation has been widely employed in array processing, system identification and spectral estimation). The received data is modeled as a linear combination of *signal vectors* from different users in additive noise. The timing estimates are formed by minimizing the projection of a given user's signal vector into the estimated noise subspace; this is known as the MUSIC algorithm [5, 6, 7].

Estimating parameters by minimizing the projection of the signal vectors into the noise subspace is equivalent to estimating these same parameters by maximizing the projection of the signal vectors into the signal subspace. Therefore, in this paper, we exploit techniques from *subspace fitting* [6, 8] to estimate the user delays. It follows that it is sufficient to estimate only the signal subspace instead of calculating the full eigenvalue or singular value decomposition. Frequently the dimension of the signal subspace is smaller than that of the noise subspace (this holds particularly in cases where a subset of users is first acquired and this information is used to acquire additional users). Furthermore, the underlying system model gives rise to a spherical noise subspace; therefore, spherical subspace methods [9, 10] can be applied which further reduce the computational complexity of the estimation process. This leads to considerable savings in terms of computation with little performance loss. The last and perhaps most important issue is the generalization to the case of random codes from bit to bit. In this case, due to the codes changing from bit to bit, the underlying subspace structures also change, making it all the more important for the signal subspace to be updated as often and accurately as possible. We briefly present a preliminary idea in this regard for tracking the delays of users. For the above reasons, we investigate strategies involving the signal subspace and its update for delay estimation.

II. SYSTEM MODEL

We assume a K-user CDMA network with all users using binary phase shift keying for both data and chip modulations. The received signal is a superposition of the K signals in additive white Gaussian noise (the non-white case can also be handled with knowledge of the

This work was supported by Nokia Inc.

[1] Refer to [4] for similar and independent work.

noise correlation matrix) given by

$$y(t) = \sum_{k=1}^{K} \alpha_k s_k(t - \tau_k) + \eta_t \qquad -\infty < t < \infty, \quad (1)$$

where α_k and τ_k are the attenuation and delay respectively corresponding to the k^{th} user and η_t is assumed to be white Gaussian noise with zero mean and a two-sided power spectral density of $\mathcal{N}_0/2$. The complex baseband representation for the k^{th} user's transmitted signal, s_k, is given by

$$s_k(t) = \sqrt{2P_k}\, e^{j\phi_k} \sum_i b_k^{(i)} a_{k,i}(t - iT), \qquad (2)$$

where P_k is the transmitted power, ϕ_k is the carrier phase relative to the local oscillator at the receiver, $b_k^{(i)} \in \{+1, -1\}$ is the i^{th} transmitted symbol, $a_{k,i}(t)$ is the spreading waveform in the i^{th} interval, and T is the symbol duration. The spreading waveform, which can potentially vary over successive bit intervals, is given by $a_{k,i}(t) = \sum_{n=0}^{N-1} \Pi_{T_c}(t - nT_c)\, a_{k,i}^{(n)}$, where $\Pi_{T_c}(t)$ is a rectangular pulse, $T_c = T/N$, and $a_{k,i}^{(n)} \in \{+1, -1\}$.

The received continuous signal is discretized and vectorized by sampling the output of a chip matched filter and buffering N samples to form one observation vector, $\mathbf{y}_i \in \mathbb{C}^N$. Since the system is asynchronous, each observation vector can be viewed as a linear combination of $2K$ signal vectors (2 components from each user) plus noise. The factors due to the power, phase, and transmitted symbols of the k^{th} user are collected into a single complex constant $c_k^{(i)}$, and the signal model for all K users can now be written as

$$\mathbf{y}_i = \sum_{k=1}^{K} \left(c_k^{(i-1)} \mathbf{u}_{k,i}^r + c_k^{(i)} \mathbf{u}_{k,i}^l \right) + \boldsymbol{\eta}_i = \mathbf{A}_i(\boldsymbol{\tau})\, \mathbf{c}_i + \boldsymbol{\eta}_i, \qquad (3)$$

where $\boldsymbol{\eta}_i = [\eta_{i,1}, \ldots, \eta_{i,N}]^\top \in \mathbb{C}^N$ is a Gaussian random vector and its elements are zero mean with variance $\sigma^2 = \frac{\mathcal{N}_0}{2T_c}$ and are mutually independent. The signal vectors $\mathbf{u}_{k,i}^r$ and $\mathbf{u}_{k,i}^l$ depend only on the k^{th} user's spreading waveforms in the $(i-1)^{th}$ and i^{th} bits and the user's delay (assuming a single path environment). If the k^{th} user's delay can be written as $\tau_k = q + \gamma$, where $q \in \{0, 1, \ldots, N-1\}$ and $\gamma \in [0, 1)$, then the corresponding signal vector can be given by

$$\begin{aligned} \mathbf{u}_{k,i}^r(\tau_k) &= \gamma a_{k,i-1}^r(q) + (1-\gamma) a_{k,i-1}^r(q+1) \\ \mathbf{u}_{k,i}^l(\tau_k) &= \gamma a_{k,i}^l(q) + (1-\gamma) a_{k,i}^l(q+1) \end{aligned} \qquad (4)$$

with $a_{k,i-1}^r(q) = [a_{k,i-1}^{(N-q)}, a_{k,i-1}^{(N-q+1)}, \ldots, a_{k,i-1}^{(N)}, 0, \ldots, 0]^\top$ and $a_{k,i}^l(q) = [0, \ldots, 0, a_{k,i}^{(0)}, \ldots, a_{k,i}^{(N-q-1)}]^\top \in \mathbb{C}^{N \times 1}$. The time index i indicates possible non-stationarity of the observation vectors $\mathbf{y}_i$ arising due to the nonrepetetive nature of the codes over multiple bit intervals. Note that in (3), we have defined $\mathbf{c}_i = [c_1^{(i-1)}\, c_1^{(i)} \cdots c_K^{(i-1)}\, c_K^{(i)}]^\top \in \mathbb{C}^{2K}$ and the signal matrix $\mathbf{A}_i = [\mathbf{u}_{1,i}^r \mathbf{u}_{1,i}^l \cdots \mathbf{u}_{K,i}^r \mathbf{u}_{K,i}^l] \in \mathbb{C}^{N \times 2K}$. In the stationary case, where the code words repeat from bit to bit, the signal vectors do not depend on i.

III. SUBSPACE AVERAGING

If we first restrict ourselves to the stationary case, the correlation matrix of the observed signal can be written as

$$\mathbf{R} = E\left[\mathbf{y}_i \mathbf{y}_i^\dagger\right] = \mathbf{A}\mathbf{C}\mathbf{A}^\dagger + \sigma^2 \mathbf{I} = \mathbf{R}^s + \sigma^2 \mathbf{I}, \qquad (5)$$

where $\mathbf{C} = E\left[\mathbf{c}_i \mathbf{c}_i^\dagger\right] \in \mathbb{C}^{2K \times 2K}$ is diagonal and σ^2 is the noise variance. Assuming the presence of K users, the correlation matrix is seen to be the sum of a rank-$2K$ matrix and a full rank noise matrix. The eigenvalue decomposition of $\mathbf{R}$ can be written as $\mathbf{R} = \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^\dagger$, where the columns of $\mathbf{V}$ are the eigenvectors of $\mathbf{R}$ and $\boldsymbol{\Lambda}$ is a diagonal matrix of the corresponding eigenvalues, λ_n, such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{2K} \geq \lambda_{2K+1} = \cdots = \lambda_N = \sigma^2$. Accordingly, $\mathbf{V}$ can be written as $\mathbf{V} = [\mathbf{E}_{\mathcal{S}}\ \mathbf{E}_{\mathcal{N}}]$, where $\mathbf{E}_{\mathcal{S}} \in \mathbb{C}^{N \times 2K}$ and $\mathbf{E}_{\mathcal{N}} \in \mathbb{C}^{N \times (N-2K)}$ span orthogonal subspaces called signal and noise subspaces respectively. One method of estimating the correlation matrix is to form a weighted average of the form

$$\hat{\mathbf{R}}_i = \sum_{j=1}^{i} \beta_j \mathbf{y}_j \mathbf{y}_j^\dagger, \qquad (6)$$

where the factor β_j is introduced as a "forgetting" factor to introduce dependency of the current correlation estimate on all the past data. This accounts for possible dependencies in the data. The standard eigenvalue decomposition is then performed on this matrix to obtain the relevant subspaces, from which we can calculate estimates of the users' delays. These operations are unfortunately too expensive to implement for updating, requiring $O(N^2 \log N)$ operations for each subspace decomposition. An important observation is the fact that in a typical cellular CDMA system, the number of users being synchronized to is typically much smaller than the spreading gain of the system. This results in a small signal subspace dimension and a large noise subspace dimension. We would therefore like to use an algorithm that is tailored to tracking only the signal subspace. One such algorithm is the spherical subspace averaging method first proposed by Karasalo [9, 10, 11].

Let us assume an estimate of the correlation matrix at the $(i-1)^{\text{th}}$ time instant, $\hat{\mathbf{R}}_{i-1} = \hat{\mathbf{R}}_{i-1}^s + \hat{\sigma}_{i-1}^2 \mathbf{I}$, where $\hat{\mathbf{R}}_{i-1}^s$ is a rank $2K$ matrix corresponding to the signal component and $\hat{\sigma}_{i-1}^2$ is the estimate of the assumed additive white Gaussian background noise. Consider an intermediate estimate of the i^{th} correlation matrix, $\hat{\mathbf{R}}_i$, of the form

$$\hat{\mathbf{R}}_i = \hat{\mathbf{R}}_i^p = \beta \hat{\mathbf{R}}_{i-1} + \mathbf{y}_i \mathbf{y}_i^\dagger. \qquad (7)$$

The estimate $\hat{\mathbf{R}}_i^p$ does not fit the model of (5) for a candidate correlation matrix for this problem, in that it cannot be written as the sum of a rank-$2K$ matrix corresponding to the signal component plus a full rank noise matrix. Being able to write $\hat{\mathbf{R}}_i$ in this form directly yields an estimate of the signal subspace, as we will see below. We therefore form an estimate $\hat{\mathbf{R}}_i = \hat{\mathbf{R}}_i^s + \hat{\sigma}_i^2\mathbf{I}$, where $\hat{\mathbf{R}}_i^s$, which is a rank-$2K$ matrix, and $\hat{\sigma}_i^2$ are chosen such that $\hat{\mathbf{R}}_i$ is the best approximation to $\hat{\mathbf{R}}_i^p$ in the Frobenius norm sense.

$$\left\{\hat{\mathbf{R}}_i^s, \hat{\sigma}_i^2\right\} = \arg\min_{\mathbf{R},\sigma^2} \left\|\mathbf{R}_i^p - \mathbf{R} - \sigma^2\mathbf{I}\right\|_F$$

It can be shown [9] that $\hat{\mathbf{R}}_i^s$ is a rank-$2K$ approximation of $\hat{\mathbf{R}}_i^p$, and $\hat{\sigma}_i^2$ is the average of the smallest $N-2K$ eigenvalues of $\hat{\mathbf{R}}_i^p$.

$$\begin{aligned} \hat{\sigma}_i^2 &= \frac{1}{N-2K}\left[\lambda_{2K+1} + (N-2K-1)\lambda_{2K+2}\right] \\ \hat{\mathbf{R}}_i^s &= \sum_{j=1}^{2K}[\lambda_j - \hat{\sigma}_j^2]\mathbf{v}_j\mathbf{v}_j^\dagger \end{aligned} \quad (8)$$

where $\hat{\mathbf{R}}_i^p = \sum_{j=1}^{N}\lambda_j\mathbf{v}_j\mathbf{v}_j^\dagger$. As seen from (8), in the process of calculating $\hat{\mathbf{R}}_i^s$ at each iteration, an orthonormal basis for the estimated signal subspace, $\hat{\mathbf{E}}_{\mathcal{S}} = \sum_{j=1}^{2K}\mathbf{v}_j\mathbf{v}_j^\dagger$, is generated. In the presence of infinitely many observations, all the noise eigenvalues would be identical and the noise subspace is termed *spherical*. Spherical subspace methods *impose* a spherical structure to the noise subspace at each iteration by averaging over all the eigenvalues corresponding to the noise subspace.

A computationally simple algorithm involving the *data* and not the correlation matrix was proposed in [9, 10]. This improves the condition of the problem and allows pipelining for real-time implementation. The incoming data vector is projected onto the estimate of the signal subspace and a *one-dimensional* representation of the orthogonal noise subspace. These projections are then used to update the signal and one-dimensional noise subspaces. The computational complexity of this algorithm can be reduced to $O(NK)$. For further details regarding spherical subspace algorithms, the reader is refered to [9, 10, 11].

It is well known that in the ideal case, the range of all the signal vectors is exactly equal to the signal subspace. In the finite observation case, however, when only an estimate of the signal subspace $\hat{\mathbf{E}}_{\mathcal{S}}$ is available, the two subspaces are not identical. Therefore we would like to find delay estimates that fit the subspace spanned by the signal vectors to the signal subspace in some "optimal" sense. Having obtained $\hat{\mathbf{E}}_{\mathcal{S}}$, the delay estimates for a particular user, $\hat{\tau}_k$ are found by maximizing the projection of its normalized signal vector, $\tilde{\mathbf{u}}_k = \tilde{\mathbf{u}}_k^r + \tilde{\mathbf{u}}_k^l$, where $\tilde{\mathbf{u}}_k = \mathbf{u}_k/\|\mathbf{u}_k\|$, into the signal subspace. We can then pose the problem as

$$\begin{aligned} \hat{\tau}_k = \arg\max_\tau f(\tau) &= \arg\max_\tau \frac{\left\|\mathbf{u}_k^\dagger(\tau)\hat{\mathbf{E}}_{\mathcal{S}}\right\|}{\|\mathbf{u}_k(\tau)\|} \\ &= \arg\max_\tau \frac{\mathbf{u}_k^\dagger(\tau)\hat{\mathbf{E}}_{\mathcal{S}}\hat{\mathbf{E}}_{\mathcal{S}}^\dagger\mathbf{u}_k(\tau)}{\mathbf{u}_k^\dagger(\tau)\mathbf{u}_k(\tau)} \end{aligned} \quad (9)$$

where $\mathbf{u}_k^\dagger(\tau)$ is as defined in (4). The function $f(\tau)$ is piecewise continuous over the N chips, and hence is maximized over each chip separately; the N maxima are compared to yield the global maximum, which yields the actual timing. Maximizing $f(\tau)$, $\tau = q+\gamma$, over one chip ($\gamma \in [0,1)$) yields a quadratic equation in γ, of the form $a\gamma^2 + b\gamma + c = 0$, where a, b and c are functions of $\{\mathbf{a}_k(q),\ \mathbf{a}_k(q+1),\ \hat{\mathbf{E}}_{\mathcal{S}}\hat{\mathbf{E}}_{\mathcal{S}}^\dagger\}$. Thus, a closed form solution for the delay which yields the maximum projection is obtained, which further decreases computation.

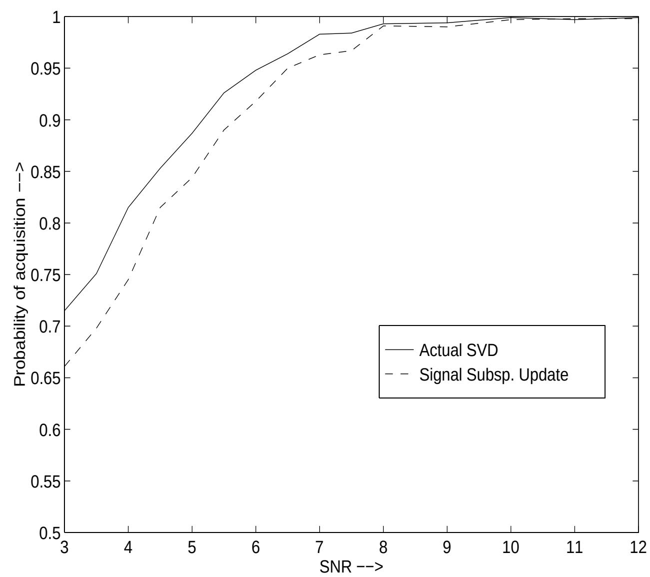

Figure 1: Probability of acquisition of the desired user as a function of SNR, using a window size of 200 bits. Spreading code length, $N = 31$, number of interfering users, $K = 6$ and their powers = 20dB.

As a comparison between the averaged signal subspace method and the standard eigenvalue decomposition, delay estimates were produced using both algorithms. A CDMA system with 6 users was considered, where the interfering users were 20dB stronger than the desired user and the spreading gain was 31 (31 chips per bit). Figures (1) and (2) depict the probability of acquisition of the full SVD and the spherical subspace updating algorithms with increasing SNR (for a fixed window size of 200 bits) and window lengths (for a fixed SNR of 6dB) respectively. As is expected, there is a direct tradeoff between the background SNR and the window length over which subspace estimation is carried out.

Due to the inherent updating structure of the spherical subspace algorithms, they are ideally suited for track-

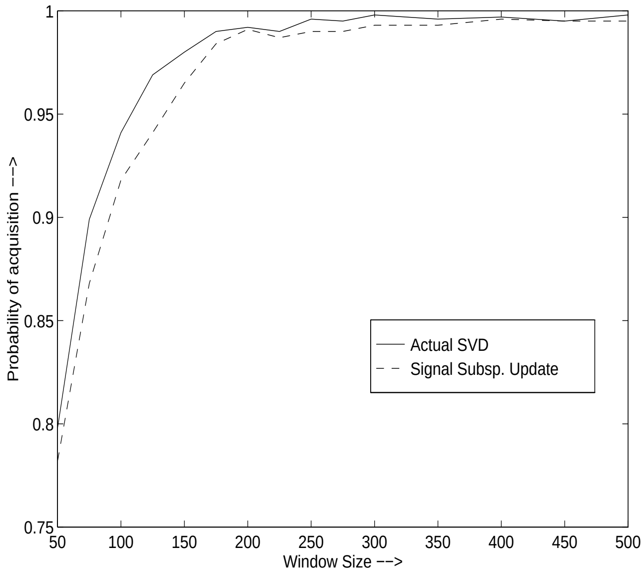

Figure 2: Probability of acquisition of the desired user as a function of window size, with a background SNR of 8dB. Spreading code length, $N = 31$, number of interfering users, $K = 6$ and their powers = 20dB.

ing data with underlying nonstationarities such as time varying delays. To examine the tracking capabilities of the subspace updating algorithm, we simulated a system where the delays of the desired user change by one chip over 2000 bits. Table (1) depicts the mean squared error, in chips, of the estimated delay calculated using both the full SVD and the signal subspace updating algorithms as a function of increasing SNR. As can be seen, the RMSEs are comparable although the computational complexity of the subspace updating strategy is considerably lesser than that of the actual SVD. For a given background SNR, the tracking capability of any algorithm based on averaged correlation estimates increases with increased window length; this is captured in Table (2).

IV. TRACKING DELAYS WITH NONREPEATING CODEWORDS

In this section we examine the feasibility of *tracking* the delays of individual users with codes that remain constant over blocks of length L and then change. The worst case scenario in this case is when $L = 1$. Ignoring the additive noise for the moment, the observation vectors $\mathbf{y}_i$ can be written as in (3)

$$\begin{aligned} \mathbf{y}_1 &= \mathbf{A}_1 \mathbf{c}_1 \\ \mathbf{y}_2 &= \mathbf{A}_2 \mathbf{c}_2 \\ &\vdots \\ \mathbf{y}_i &= \mathbf{A}_i \mathbf{c}_i \end{aligned} \tag{10}$$

As can be seen, the different $\mathbf{y}_i$'s originate from different underlying subspaces, $\mathbf{A}_i$'s. Hence, a direct subspace decomposition of the correlation matrix estimate, $\hat{\mathbf{R}}_i$,

SNR (dB)	Mean Sq. Error (in chips)	
	Full SVD	Spher. Subsp.
4	1.129×10^{-1}	1.430×10^{-1}
5	9.564×10^{-2}	1.126×10^{-1}
6	6.684×10^{-2}	8.932×10^{-2}
7	6.648×10^{-2}	6.756×10^{-2}
8	4.747×10^{-2}	5.185×10^{-2}
9	2.753×10^{-2}	3.020×10^{-2}
10	2.089×10^{-2}	2.217×10^{-2}

Table 1: Mean squared tracking error (in chips) compared using the full SVD and the spherical subspace method for different values of SNR. The delay is varied at a rate of 1 chip in 2000 bits.

Window Length	Mean Sq. Error (in chips)	
	Full SVD	Spher. Subsp.
50	9.956×10^{-2}	1.003×10^{-1}
100	7.751×10^{-2}	8.925×10^{-2}
150	3.788×10^{-2}	3.701×10^{-2}
200	3.466×10^{-2}	3.734×10^{-2}
250	2.346×10^{-2}	2.932×10^{-2}
300	1.929×10^{-2}	2.030×10^{-2}
350	1.021×10^{-2}	1.460×10^{-2}
400	5.983×10^{-3}	7.683×10^{-3}

Table 2: Mean squared tracking error (in chips) compared using the full SVD and the spherical subspace method for different window lengths. The delay is again assumed to vary at a rate of 1 chip in 2000 bits.

as given in (6) cannot be calculated. We therefore seek to transform the i^{th} observation vector $\mathbf{y}_i$ by a linear transformation $\mathbf{T}_i$ such that $\mathbf{T}_i\mathbf{A}_i = \mathbf{A}_1$. Then each equation in the set (10) gets transformed to

$$\mathbf{z}_i = \mathbf{T}_i\mathbf{y}_i = \mathbf{T}_i\mathbf{A}_i\,\mathbf{c}_i = \mathbf{A}_1\mathbf{c}_i\ .$$

Now we can form the correlation matrix using the $\mathbf{z}_i$'s if we know the $\mathbf{T}_i$'s, and the spherical subspace method discussed in Section III can be employed to obtain delay estimates. Unfortunately, $\mathbf{T}_i$ is a function of the delays $\boldsymbol{\tau}$. To get a good initial estimate of the $\mathbf{T}_i$'s a preamble of L_1 bits is sent with a fixed spreading code modulated onto each bit. The delay of each user is acquired to within a chip in the preamble, after which the spreading codes for each user are randomized from bit to bit. Table (3) depicts the mean squared error in tracking the estimated timing for the desired

SNR (dB)	Mean Sq. Error (in chips)
3	3.340×10^{-2}
4	2.912×10^{-2}
5	2.600×10^{-2}
6	3.235×10^{-2}
7	3.620×10^{-2}
8	3.775×10^{-2}
9	2.204×10^{-2}
10	2.293×10^{-2}
11	2.342×10^{-2}

Table 3: Mean squared tracking error (in chips) using the spherical subspace method for different values of SNR. Initial estimate is assumed to be within a chip of true delay.

user with increasing signal-to-noise ratio (SNR). In our study, we have assumed a stationary, white structure on the noise, whereas, in reality, the noise covariance structure changes from bit to bit due to the transformation. It is therefore desirable to select nearly orthogonal transformations $\mathbf{T}_j$ that only "approximately" solve the equation $\mathbf{T}_i\mathbf{A}_i \approx \mathbf{A}_1$. We are still investigating this possibility in conjunction with spreading code design to realize this transformation.

V. CONCLUSIONS

We considered the problem of acquiring the delays of a certain group of mobiles in a CDMA cellular communication system. A subspace based approach was adopted due to its near-far resistance properties and relative computational simplicity compared to other near-far algorithms (like ML methods). In most situations, the users being acquired span a small dimensional subspace of the space of observations. Hence, it makes sense to compute the signal subspace rather than the noise subspace or both. For this reason, spherical subspace algorithms were examined for delay estimation in a multiuser CDMA communication system. Since they only update the signal subspace, these signal subspace updating strategies are computationally less expensive than the full SVD, requiring only $O(NK)$ operations per update, where N is the spreading gain and K, the number of users being acquired. In addition, the inherent updating structure of these algorithms accounts for nonstationarities arising from time varying delays; hence they are also suitable for delay tracking. Our studies, manifest in the plots in Section III, indicate that the performance of this class of algorithms is comparable to actual SVD calculation. Hence they form a very viable set of algorithms for future use and study.

VI. REFERENCES

[1] S. Y. Miller and S. C. Schwartz, "Parameter estimation for asynchronous multiuser communication," in *Proceedings of the 1989 Conference on Information Sciences and Systems*, pp. 294–299, 1989.

[2] U. Fawer and B. Aazhang, "A multiuser receiver for code division multiple access communications over multipath channels," in *Proceedings of GLOBCOM (Communication Theory Mini-Conference)*, Dec. 1992.

[3] S. E. Bensley and B. Aazhang, "Subspace-based estimation of multipath channel parameters for code division multiple access communication systems," in *GLOBCOM*, December, 1994.

[4] E. G. Ström, S. Parkvall, and B. E. Ottersten, "Near-far resistant propagation delay estimators for asynchronous direct-sequence code division multiple access systems," in *Proceedings of the 1994 International Zurich Seminar on Digital Communications*, pp. 251–260, Springer Lecture Notes in Computer Science 783, Mar. 1994.

[5] D. H. Johnson and D. E. Dudgeon, *Array Signal Processing: Concepts and Techniques*. Englewood Cliffs, NJ: Prentice Hall, 1993.

[6] A. Van der Veen, E. Deprettere, and A. Swindlehurst, "Subspace-based signal analysis using singular value decomposition," *Proceedings of the IEEE*, vol. 81, pp. 1277–1308, Sept. 1993.

[7] P. Stoica and A. Nehorai, "Performance comparison of subspace rotation and music methods for direction estimation," *IEEE Trans. Signal Processing*, vol. 39, pp. 446–453, 1991.

[8] B. Ottersten, M. Viberg, P. Stoica, and A. Nehorai, "Exact and large sample maximum likelihood techniques for parameter estimation and detection in array processing," in *Radar Array Processing* (T. J. S. S. haykin, J. Litva, ed.), vol. 25 of *Springer Series in Information Sciences*, Springer-Verlag, 1993.

[9] I. Karasalo, "Estimating the covariance matrix by signal subspace averaging," *IEEE Trans. on Acoustics, Speech, and Signal Processing*, vol. ASSP-34, pp. 8–12, Feb. 1986.

[10] A. Kavčić and B. Yang, "A new efficient subspace tracking algorithm based on singular value decomposition," in *Proceedings of ICASSP*, vol. 4, pp. 485–488, 1994.

[11] P. Comon and G. H. Golub, "Tracking a few extreme singular values and vectors in signal processing," *Proceedings of the IEEE*, vol. 78, Aug. 1990.